Estimating Symmetric, Positive Definite Matrices in Robotic Control

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Abstract— **In a number of contexts relevant to control problems, including estimation of robot dynamics, covariance, and smart structure mass and stiffness matrices, we need to solve an over-determined set of linear equations $\mathbf{AX} \approx \mathbf{B}$ with the constraint that the matrix X be symmetric and positive definite. In the classical least squares method, the measurements of A are assumed to be free of error. Hence, all errors are confined to B. Thus, the "optimal" solution is given by minimizing $\|\mathbf{AX} - \mathbf{B}\|_F^2$. However, this assumption is often impractical. Sampling errors, modeling errors, and, sometimes, human errors bring inaccuracies to A as well. In this paper, we introduce a different optimization criterion, based on area, which takes the errors in both A and B into consideration. The analytic expression of the global optimizer is derived. The algorithm is applied to identify the joint space mass-inertia matrix of a Gough-Stewart platform. Experimental results indicate that the new approach is practical, and improves performance.**

Keywords— **Mass-inertia estimation, covariance estimation, stiffness estimation, symmetric positive definite matrix.**

I. Introduction

ESTIMATION of symmetric positive definite matrices is required when solving a variety of control problems including robotic control, smart structure control, and intelligent control. In robotics, the mass-inertia matrix of a robotic system is in the symmetric positive definite class, and the accuracy of its estimate directly affects control performance [1][2][3]. Similarly, controlling vibrations and precise positions of "smart" structures often requires estimation of the structure's mass and stiffness matrices [4][5]; both are symmetric and positive definite. In intelligent control, control decisions are often made based on estimation of a covariance matrix [6][7][8], which is, of course, symmetric and positive definite. Estimation of symmetric positive definite matrices also appears, to a lesser extent, in fields outside control including the educational testing [9] and matrix modification problems [10]. Most of the above examples can be formulated directly or indirectly into finding an optimal solution of a set of linear equations

$$\mathbf{AX} \approx \mathbf{B} \tag{1}$$

where $\mathbf{A}, \mathbf{B} \in \mathcal{R}^{m\times n}$ are given, $\mathbf{X} \in \mathcal{P}$ is the fitting matrix, $\mathcal{P}$ is the set of symmetric and positive definite matrices with size $n \times n$. For example, the estimation of the joint space mass-inertia matrix of a flexure jointed hexapod (Stewart Platform) [3] and stiffness matrix directly fit into (1). The covariance matrix estimation problem and the matrix modification problem (with symmetric positive definite constraint) can be regarded as extracting a symmetric positive definite matrix ($\mathbf{C}^*$) from a symmetric but indefinite matrix ($\mathbf{C}$). Thus it can be formulated as solving $\mathbf{CX} \approx \mathbf{I}$ where $\mathbf{X} \in \mathcal{P}$, $\mathbf{I}$ being the identity matrix of size $n\times n$. The "optimal" $\mathbf{C}^*$ is given by $\mathbf{C}^* = \mathbf{X}^{-1}$.

There is a rich resource of prior work on this type of problem. Space limitations do not allow us to present a broad survey. Instead we try to emphasize some of the work that is most related to our work. Higham [11] finds an optimal symmetric estimate using the least squares approach (Symmetric Procrustes Problem). Although the positive definite constraint is not considered in his method, Higham shows that the estimate will be positive (semi-)definite if the data matrix $\mathbf{A}^T\mathbf{B} + \mathbf{B}^T\mathbf{A}$ is positive (semi-)definite. Nothing can be concluded about the definiteness of the estimate if $\mathbf{A}^T\mathbf{B} + \mathbf{B}^T\mathbf{A}$ is indefinite. Hu [12] presents a least squares based method to handle the positive definite constraint. In his method, the upper and lower bounds for each entry of the fitting matrix must be given explicitly as the constraint. A non-negative scalar is also introduced as a constraint, which measures the degree of positive definiteness. Using the least squares criterion, $\|\mathbf{AX} - \mathbf{B}\|_F^2$ where $\|\cdot\|_F$ denotes the Frobenius norm of a matrix, the problem can also be cast as a semi-definite program [13] by specifying lower (and/or upper) bounds of the eigenvalues of $\mathbf{X}$.

Nevertheless, in many applications, there is a question of the suitability of the least squares criterion $\|\mathbf{AX} - \mathbf{B}\|_F^2$. In the classical least squares approach,

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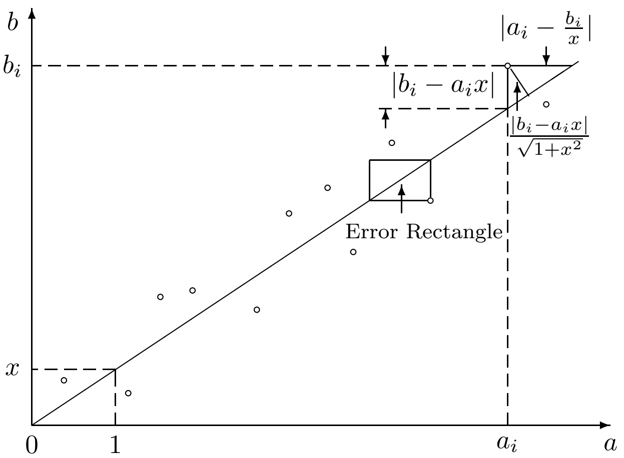

Fig. 1. Geometric interpretations of one parameter estimation using the least squares, the total least squares, and the new approaches.

the measurements $\mathbf{A}$ are supposed to be free of error, hence, all errors are restricted to $\mathbf{B}$. However, this assumption is frequently impractical. Sampling errors, modeling errors, and, sometimes, human errors bring inaccuracies to $\mathbf{A}$ as well. For example, in the estimation of a flexure jointed hexapod's joint space mass-inertia matrix [3], $\mathbf{A}$ and $\mathbf{B}$ contain the measurements of payload accelerations and base forces, respectively. As a result, sampling and instrument noises appear in both $\mathbf{A}$ and $\mathbf{B}$. Similar phenomenon happens in identifying a robot dynamic model [14]. Thus, it is natural for one to expect improved performance by employing a criterion that is capable of describing the errors occurring in both measurement matrices, rather than using the least squares criterion in which only the errors in $\mathbf{B}$ are considered.

In this paper, we present a new method of solving an over-determined set of linear equations (1) with $\mathbf{X}$ being symmetric positive definite, and both $\mathbf{A}$ and $\mathbf{B}$ containing errors.

II. Problem Formulation

A simple example will be more intuitive than a complex one for illustrating and understanding the motivation for the new optimization criterion. So let's consider the following problem with only one variable: estimating a single parameter from a set of over-determined equations

$$\vec{a}x \approx \vec{b}$$

where $\vec{a} = [a_1, a_2, \cdots, a_m]^T$, $\vec{b} = [b_1, b_2, \cdots, b_m]^T \in \mathcal{R}^m$ are known data vectors, $x \in \mathcal{R}$ is the variable to be estimated. Using the classical least squares approach, the solution is the minimizer of the optimization criterion $(\vec{b}-\vec{a}x)^T(\vec{b}-\vec{a}x)$ or equivalently $\sum_{i=1}^{m}(b_i - a_i x)^2$. Geometrically, as shown in Figure 1, this criterion is the summation of the squared vertical "errors" (the distance from a data point (a_i, b_i) to the fitting line along the direction of b axis). This criterion is reasonable if the errors only occur in the data vector $\vec{b}$, because we are making predictions based on $\vec{a}$ that is free of error. If the errors are confined to $\vec{a}$, and $\vec{b}$ is free of error, the least squares approach is still appropriate, because we can minimize $(\vec{a} - \frac{\vec{b}}{x})^T(\vec{a} - \frac{\vec{b}}{x})$ or equivalently $\sum_{i=1}^{m}(a_i - \frac{b_i}{x})^2$, which will give the estimate of $\frac{1}{x}$. As shown in Figure 1, this time, the least squares solution minimizes the summation of the squared horizontal "errors" (the distance from a data point to the fitting line along the direction of a axis).

However, in many applications, both $\vec{a}$ and $\vec{b}$ are measurements containing errors. Under this scenario, a more appropriate approach of fitting is the total least squares method [15] (termed orthogonal regression or errors-in-variables regression in the statistical literature). For the above single parameter estimation problem, the total least squares solution minimizes $\sum_{i=1}^{m}\frac{(b_i - a_i x)^2}{(1+x^2)}$, which, as shown in Figure 1, is the summation of the squared minimum "errors" (the minimum distance from a data point to the fitting line) [1]. From the properties of the right triangle we can easily derive $\frac{(b_i-a_ix)^2}{(1+x^2)} = \frac{(b_i-a_ix)^2(a_i-\frac{b_i}{x})^2}{(b_i-a_ix)^2+(a_i-\frac{b_i}{x})^2}$, i.e., the minimum "error" contains the information of both the vertical "error" and the horizontal "error".

Motivated by above geometric interpretations of the least squares and the total least squares methods, we introduce a new optimization criterion, the area criterion, which is defined as the summation of the areas of the "error rectangles", i.e., $\sum_{i=1}^{m}|b_i - a_i x||a_i - \frac{b_i}{x}|$. As shown in Figure 1, the ith "error rectangle" is constructed by the ith vertical and ith horizontal "errors". Considering the symmetric and positive definite constraints (in this example, it implies $x > 0$), the area criterion can be equivalently written as

$$\sum_{i=1}^{m}|b_i - a_i x|\left|a_i - \frac{b_i}{x}\right| = \left\|\vec{a}y - \frac{\vec{b}}{y}\right\|_2^2$$

where $y \in \mathcal{R}$, $y \neq 0$, $x = yy^T = y^2$. Note that we have transformed the positive constraint on x to the invertible constraint on y.

Now let's consider the original problem given by (1). The area criterion is then extended as $\mathrm{Tr}[(\mathbf{AX} - \mathbf{B})^T(\mathbf{A} - \mathbf{BX}^{-1})]$ where Tr stands for trace, $\mathbf{AX} - \mathbf{B}$ represents the errors in $\mathbf{B}$ from the predictions based on $\mathbf{A}$, and $\mathbf{A} - \mathbf{BX}^{-1}$ represents the errors in $\mathbf{A}$ from

[1]To our knowledge, employing the symmetric and positive definite matrix constraints in the total least squares method is still an open problem.

the predictions based on $\mathbf{B}$. Using the properties of matrix calculus and the fact that $\mathbf{X} = \mathbf{Y}\mathbf{Y}^T$ for any $\mathbf{X} \in \mathcal{P}$ where $\mathbf{Y} \in \mathcal{I}$, $\mathcal{I}$ being the set of real invertible matrices, the above extended area criterion can be equivalently written as $\|\mathbf{A}\mathbf{Y} - \mathbf{B}\mathbf{Y}^{-T}\|_F^2$. Thus, we can define an optimization problem as follows.

Definition II.1: (Symmetric Positive Definite Estimation problem, SPDE) For an over-determined set of m linear equations $\mathbf{A}\mathbf{X} \approx \mathbf{B}$, where $\mathbf{A}, \mathbf{B} \in \mathcal{R}^{m\times n}$ are given, $\mathbf{X} \in \mathcal{P}$ is the fitting matrix, let the area criterion, $f : \mathcal{I} \to \mathcal{R}$, be defined as

$$f(\mathbf{Y}) = \|\mathbf{A}\mathbf{Y} - \mathbf{B}\mathbf{Y}^{-T}\|_F^2.$$

The symmetric positive definite estimate $\mathbf{X}^*$ is given by $\mathbf{X}^* = \mathbf{Y}^*\mathbf{Y}^{*T}$ where $\mathbf{Y}^*$ is a minimizer of $f(\mathbf{Y})$.

III. Finding The Optimizer

To simplify derivations, we introduce two optimization criteria which differ from $f(\mathbf{Y})$ by only a constant.

Lemma III.1: Let $g : \mathcal{I} \to \mathcal{R}$ and $h : \mathcal{P} \to \mathcal{R}$ be defined by

$$\begin{aligned} g(\mathbf{Y}) &= \mathrm{Tr}(\mathbf{Y}^T\mathbf{P}\mathbf{Y} + \mathbf{Y}^{-1}\mathbf{Q}\mathbf{Y}^{-T}), && (2)\\ h(\mathbf{X}) &= \mathrm{Tr}(\mathbf{P}\mathbf{X} + \mathbf{X}^{-1}\mathbf{Q}) && (3) \end{aligned}$$

where $\mathbf{P} = \mathbf{A}^T\mathbf{A}$ and $\mathbf{Q} = \mathbf{B}^T\mathbf{B}$. Then minimizing $f(\mathbf{Y})$ on $\mathcal{I}$, minimizing $g(\mathbf{Y})$ on $\mathcal{I}$, and minimizing $h(\mathbf{X})$ on $\mathcal{P}$ are equivalent, i.e., $\mathbf{Y}^* \in \mathcal{I}$ minimizes $f(\mathbf{Y})$ if and only if $\mathbf{Y}^*$ minimizes $g(\mathbf{Y})$ if and only if $\mathbf{X}^* = \mathbf{Y}^*\mathbf{Y}^{*T} \in \mathcal{P}$ minimizes $h(\mathbf{X})$.

Proof: From the identities in matrix calculus, we have

$$f(\mathbf{Y}) = g(\mathbf{Y}) - 2\,\mathrm{Tr}(\mathbf{A}^T\mathbf{B}) = h(\mathbf{X}) - 2\,\mathrm{Tr}(\mathbf{A}^T\mathbf{B}) \quad (4)$$

where $\mathbf{X} = \mathbf{Y}\mathbf{Y}^T$. $\square$

In the following two theorems, we assume that $\mathrm{Rank}(\mathbf{A}) = \mathrm{Rank}(\mathbf{B}) = n$, i.e., $\mathbf{P}, \mathbf{Q} \in \mathcal{P}$. This assumption is easy to satisfy in most applications. At the end of this section, we will show that with only minor modification the results can be easily extended to the case that $\mathbf{A}$ is full rank and $\mathbf{B}$ loses rank.

Lemma III.1 implies that it is sufficient to derive the normal equation for one of the optimization criteria ($f(\mathbf{Y})$, $g(\mathbf{Y})$, or $h(\mathbf{X})$). We derive the normal equation for $g(\mathbf{Y})$ as follows.

Theorem III.2: Let $g(\mathbf{Y})$ be defined by (2). If $\mathbf{Y}^*$ is a minimizer of $g(\mathbf{Y})$, then it satisfies

$$\mathbf{Y}^*\mathbf{Y}^{*T}\mathbf{P}\mathbf{Y}^*\mathbf{Y}^{*T} = \mathbf{Q}. \quad (5)$$

Proof: Let $g' : \mathcal{X} \times \mathcal{X} \to \mathcal{R}$ be defined as

$$g'(\mathbf{Y}, \mathbf{Z}) = \mathrm{Tr}(\mathbf{Y}^T\mathbf{P}\mathbf{Y} + \mathbf{Z}\mathbf{Q}\mathbf{Z}^T)$$

where $\mathcal{X}$ is the set of real $n \times n$ matrices. Then minimizing $g(\mathbf{Y})$ on $\mathcal{I}$ is equivalent to minimizing $g'(\mathbf{Y}, \mathbf{Z})$ on $\mathcal{X} \times \mathcal{X}$ with the constraint $\mathbf{Y}\mathbf{Z} = \mathbf{I}$ where $\mathbf{I} \in \mathcal{R}^{n\times n}$ is the identity matrix.

Let $\mathbf{Y}, \mathbf{Z}, \mathbf{\Psi} \in \mathcal{X}$, $\mathbf{Y} = [\vec{y}_1, \vec{y}_2, \cdots, \vec{y}_n]^T$, and $\mathbf{Z} = [\vec{z}_1, \vec{z}_2, \cdots, \vec{z}_n]$. Let y_{ij}, z_{ij}, and ψ_{ij} be the ijth entries of $\mathbf{Y}$, $\mathbf{Z}$, and $\mathbf{\Psi}$, respectively. The Lagrangian, $L : \mathcal{X}\times\mathcal{X}\times\mathcal{X} \to \mathcal{R}$, associated with the constraint $\mathbf{Y}\mathbf{Z} = \mathbf{I}$ is defined as

$$L(\mathbf{Y}, \mathbf{Z}, \mathbf{\Psi}) = \mathrm{Tr}[\mathbf{Y}^T\mathbf{P}\mathbf{Y} + \mathbf{Z}\mathbf{Q}\mathbf{Z}^T + \mathbf{\Psi}(\mathbf{Y}\mathbf{Z} - \mathbf{I})].$$

Setting the partial derivatives of L with respect to y_{ij}, z_{ij}, and ψ_{ij} to 0's for all $1 \le i, j \le n$ gives,

$$\begin{aligned} 2\mathbf{Y}^T\mathbf{P} + \mathbf{Z}\mathbf{\Psi} &= \mathbf{0}, && (6)\\ 2\mathbf{Q}\mathbf{Z}^T + \mathbf{\Psi}\mathbf{Y} &= \mathbf{0}, && (7)\\ \mathbf{Y}\mathbf{Z} &= \mathbf{I}. && (8) \end{aligned}$$

Solving (6-8) for $\mathbf{Y}$ gives (5). $\square$

Theorem III.2 and Lemma III.1 imply two facts:
1. Any symmetric and positive definite estimate, $\mathbf{X}^*$, of the SPDE problem must satisfy

$$\mathbf{X}^*\mathbf{P}\mathbf{X}^* = \mathbf{Q} \quad (9)$$

where $\mathbf{X}^* = \mathbf{Y}^*\mathbf{Y}^{*T}$, $\mathbf{Y}^*$ is a solution of (5).
2. Any minimizer for (3) must also satisfy (9).
However, we still need to show that the solutions (or a solution) of (5) minimize(s) (2). From Lemma III.1 and the above facts, this is equivalent to verifying that the solutions (or a solution) of (9) minimize(s) (3), which is proven in the following theorem.

Theorem III.3: The unique minimizer of (3), which is the unique solution of (9), is given by

$$\mathbf{X}^* = \mathbf{U_P}\mathbf{\Sigma_P}^{-1}\mathbf{U_{\tilde{Q}}}\mathbf{\Sigma_{\tilde{Q}}}\mathbf{U_{\tilde{Q}}}^T\mathbf{\Sigma_P}^{-1}\mathbf{U_P}^T \quad (10)$$

where

$$\begin{aligned} \mathbf{P} &= \mathbf{U_P}\mathbf{\Sigma_P}^2\mathbf{U_P}^T, && (11)\\ \tilde{\mathbf{Q}} &= \mathbf{\Sigma_P}\mathbf{U_P}^T\mathbf{Q}\mathbf{U_P}\mathbf{\Sigma_P} = \mathbf{U_{\tilde{Q}}}\mathbf{\Sigma_{\tilde{Q}}}^2\mathbf{U_{\tilde{Q}}}^T && (12) \end{aligned}$$

are the Schur decomposition of $\mathbf{P}$ and $\tilde{\mathbf{Q}}$ respectively, and

$$\begin{aligned} \mathbf{\Sigma_P} &= diag[\sqrt{\lambda^1_{\mathbf{P}}}, \sqrt{\lambda^2_{\mathbf{P}}}, \cdots, \sqrt{\lambda^n_{\mathbf{P}}}],\\ \mathbf{\Sigma_{\tilde{Q}}} &= diag[\sqrt{\lambda^1_{\tilde{\mathbf{Q}}}}, \sqrt{\lambda^1_{\tilde{\mathbf{Q}}}}, \cdots, \sqrt{\lambda^n_{\tilde{\mathbf{Q}}}}] \end{aligned}$$

where $\lambda^i_{\mathbf{P}}$'s and $\lambda^j_{\tilde{\mathbf{Q}}}$'s are the ith and jth eigenvalues of $\mathbf{P}$ and $\tilde{\mathbf{Q}}$, respectively.

Proof: Substituting (11) into (9) gives

$$\mathbf{X}^*\mathbf{U_P}\mathbf{\Sigma_P}\mathbf{U_P}^T\mathbf{U_P}\mathbf{\Sigma_P}\mathbf{U_P}^T\mathbf{X}^* = \mathbf{Q}. \quad (13)$$

Left multiplying both sides of (13) by $\mathbf{\Sigma_P}\mathbf{U_P}^T$, right multiplying both sides of (13) by $\mathbf{U_P}\mathbf{\Sigma_P}$, substituting (12) into (13), and collecting terms, we have

$$(\mathbf{\Sigma_P}\mathbf{U_P}^T\mathbf{X}^*\mathbf{U_P}\mathbf{\Sigma_P})^2 = (\mathbf{U_{\tilde{Q}}}\mathbf{\Sigma_{\tilde{Q}}}\mathbf{U_{\tilde{Q}}}^T)^2.$$

It is clear that $(\mathbf{U}_{\tilde{\mathbf{Q}}}\mathbf{\Sigma}_{\tilde{\mathbf{Q}}}\mathbf{U}_{\tilde{\mathbf{Q}}}^T)^2 \in \mathcal{P}$, and $\mathbf{X}^* \in \mathcal{P}$ if and only if $\mathbf{\Sigma_P U_P^T X^* U_P \Sigma_P} \in \mathcal{P}$. Since a symmetric positive definite matrix has a unique symmetric positive definite square root, we have

$$\mathbf{\Sigma_P U_P^T X^* U_P \Sigma_P} = \mathbf{U}_{\tilde{\mathbf{Q}}}\mathbf{\Sigma}_{\tilde{\mathbf{Q}}}\mathbf{U}_{\tilde{\mathbf{Q}}}^T. \qquad (14)$$

Solving (14) gives (10).

Next we will show that the $\mathbf{X}^*$ given by (10) minimizes $h(\mathbf{X})$. Let the Schur decomposition of $\mathbf{X} \in \mathcal{P}$ be

$$\mathbf{X} = \mathbf{U_X \Sigma_X^2 U_X^T} \qquad (15)$$

where $\mathbf{\Sigma_X} = diag[\sqrt{\lambda^1_{\mathbf{X}}}, \sqrt{\lambda^2_{\mathbf{X}}}, \cdots, \sqrt{\lambda^n_{\mathbf{X}}}]$ with $\lambda^i_{\mathbf{X}}$ being the ith eigenvalue of $\mathbf{X}$. Equation (12) can be written as

$$\mathbf{Q} = \mathbf{U_P \Sigma_P^{-1} U}_{\tilde{\mathbf{Q}}}\mathbf{\Sigma}_{\tilde{\mathbf{Q}}}\mathbf{U}_{\tilde{\mathbf{Q}}}^T\mathbf{U}_{\tilde{\mathbf{Q}}}\mathbf{\Sigma}_{\tilde{\mathbf{Q}}}\mathbf{U}_{\tilde{\mathbf{Q}}}^T\mathbf{\Sigma_P^{-1} U_P^T}. \qquad (16)$$

Substituting equations (11,15,16) into (3), we have

$$\begin{aligned}
h(\mathbf{X}) &= \mathrm{Tr}(\mathbf{U_P \Sigma_P^2 U_P^T U_X \Sigma_X^2 U_X^T} + \mathbf{U_P \Sigma_P^{-1} U}_{\tilde{\mathbf{Q}}} \\
&\quad \mathbf{\Sigma}_{\tilde{\mathbf{Q}}}\mathbf{U}_{\tilde{\mathbf{Q}}}^T\mathbf{U}_{\tilde{\mathbf{Q}}}\mathbf{\Sigma}_{\tilde{\mathbf{Q}}}\mathbf{U}_{\tilde{\mathbf{Q}}}^T\mathbf{\Sigma_P^{-1} U_P^T U_X \Sigma_X^{-2} U_X^T}) \\
&= \mathrm{Tr}[(\mathbf{\Sigma_P U_P^T U_X \Sigma_X})(\mathbf{\Sigma_X U_X^T U_P \Sigma_P}) + \\
&\quad (\mathbf{U}_{\tilde{\mathbf{Q}}}\mathbf{\Sigma}_{\tilde{\mathbf{Q}}}\mathbf{U}_{\tilde{\mathbf{Q}}}^T\mathbf{\Sigma_P^{-1} U_P^T U_X \Sigma_X^{-1}})(\mathbf{\Sigma_X^{-1} U_X^T} \\
&\quad \mathbf{U_P \Sigma_P^{-1} U}_{\tilde{\mathbf{Q}}}\mathbf{\Sigma}_{\tilde{\mathbf{Q}}}\mathbf{U}_{\tilde{\mathbf{Q}}}^T)] \\
&= \mathrm{Tr}[(\mathbf{\Sigma_P U_P^T U_X \Sigma_X} - \mathbf{U}_{\tilde{\mathbf{Q}}}\mathbf{\Sigma}_{\tilde{\mathbf{Q}}}\mathbf{U}_{\tilde{\mathbf{Q}}}^T\mathbf{\Sigma_P^{-1} U_P^T} \\
&\quad \mathbf{U_X \Sigma_X^{-1}})(\mathbf{\Sigma_P U_P^T U_X \Sigma_X} - \mathbf{U}_{\tilde{\mathbf{Q}}}\mathbf{\Sigma}_{\tilde{\mathbf{Q}}}\mathbf{U}_{\tilde{\mathbf{Q}}}^T\mathbf{\Sigma_P^{-1}} \\
&\quad \mathbf{U_P^T U_X \Sigma_X^{-1}})^T + \mathbf{\Sigma_P U_P^T U_X \Sigma_X \Sigma_X^{-1} U_X^T} \\
&\quad \mathbf{U_P \Sigma_P^{-1} U}_{\tilde{\mathbf{Q}}}\mathbf{\Sigma}_{\tilde{\mathbf{Q}}}\mathbf{U}_{\tilde{\mathbf{Q}}}^T + \mathbf{U}_{\tilde{\mathbf{Q}}}\mathbf{\Sigma}_{\tilde{\mathbf{Q}}}\mathbf{U}_{\tilde{\mathbf{Q}}}^T\mathbf{\Sigma_P^{-1} U_P^T} \\
&\quad \mathbf{U_X \Sigma_X^{-1} \Sigma_X U_X^T U_P \Sigma_P}] \\
&= \|\mathbf{\Sigma_P U_P^T U_X \Sigma_X} - \mathbf{U}_{\tilde{\mathbf{Q}}}\mathbf{\Sigma}_{\tilde{\mathbf{Q}}}\mathbf{U}_{\tilde{\mathbf{Q}}}^T\mathbf{\Sigma_P^{-1} U_P^T U_X} \\
&\quad \mathbf{\Sigma_X^{-1}}\|_F^2 + 2\,Tr(\mathbf{\Sigma}_{\tilde{\mathbf{Q}}}). \qquad (17)
\end{aligned}$$

Clearly, $h(\mathbf{X})$ achieves the global minimum when

$$\mathbf{\Sigma_P U_P^T U_X \Sigma_X} = \mathbf{U}_{\tilde{\mathbf{Q}}}\mathbf{\Sigma}_{\tilde{\mathbf{Q}}}\mathbf{U}_{\tilde{\mathbf{Q}}}^T\mathbf{\Sigma_P^{-1} U_P^T U_X \Sigma_X^{-1}}, \qquad (18)$$

and $\mathbf{X}^*$ is the only solution to (18). □

Corollary III.4: The symmetric positive definite estimate, $\mathbf{X}^*$, of the SPDE problem is given by equation (10). The minimum of the area criterion, $f(\mathbf{Y})$, is $2\,\mathrm{Tr}(\mathbf{\Sigma}_{\tilde{\mathbf{Q}}} - \mathbf{A}^T\mathbf{B})$.

Proof: It follows directly from the Definition II.1 and equations (4,17). □

Remark III.5: Actually, the set of linear equations (1) to be solved need not be over-determined. All the above results still hold when $m = n$ provided that $\mathrm{Rank}(\mathbf{A}) = \mathrm{Rank}(\mathbf{B}) = n$.

Remark III.6: Theorem III.3 says that $h(\mathbf{X})$ has a unique minimizer on $\mathcal{P}$. But the minimizers of $f(\mathbf{Y})$ or $g(\mathbf{Y})$ on $\mathcal{I}$ are not unique. In fact, it is easy to show that if $\mathbf{Y}^* \in \mathcal{I}$ is a minimizer of $f(\mathbf{Y})$ or $g(\mathbf{Y})$ then $\mathbf{Y}^*\mathbf{U}$ is also a minimizer of $f(\mathbf{Y})$ and $g(\mathbf{Y})$ for any orthonormal matrix $\mathbf{U}$. Thus $f(\mathbf{Y})$ and $g(\mathbf{Y})$ have infinitely many minimizers on $\mathcal{I}$. Moreover, all these minimizers are related to the unique minimizer, $\mathbf{X}^*$, of $h(\mathbf{X})$ on $\mathcal{P}$ by $\mathbf{Y}^*\mathbf{Y}^{*T} = \mathbf{X}^*$. Consequently, the symmetric positive definite estimate is unique.

In the above discussions the data matrices $\mathbf{A}$ and $\mathbf{B}$ are assumed to be full rank. If either $\mathbf{A}$ or $\mathbf{B}$ lose rank, the method described above can not produce a symmetric positive definite optimizer. However, if $\mathbf{B}$ loses rank and $\mathbf{A}$ remains full rank, i.e., $\mathbf{P} \in \mathcal{P}$ and $\mathbf{Q} \in \overline{\mathcal{P}}$ (the set of symmetric positive semi-definite matrices), we can still find a positive semi-definite optimizer provided that $h(\mathbf{X})$ is optimized on $\overline{\mathcal{P}}_{Rank(\mathbf{Q})}$ (the set of symmetric positive semi-definite matrices with rank equal to the rank of $\mathbf{Q}$), and $\mathbf{X}^{-1}$ in $h(\mathbf{X})$ is replaced by $\mathbf{X}^+$ (Moore-Penrose pseudo-inverse of $\mathbf{X}$). The result is given as follows.

Corollary III.7: If $\mathbf{P} \in \mathcal{P}$, $\mathbf{Q} \in \overline{\mathcal{P}}$, $Rank(\mathbf{Q}) = r$, then the $\mathbf{X}^* \in \overline{\mathcal{P}}_r$ given by equation (10) minimizes the optimality criterion $\overline{h}(\mathbf{X}) = Tr(\mathbf{PX} + \mathbf{X}^+\mathbf{Q})$. The global minimum of $\overline{h}(\mathbf{X})$ on $\overline{\mathcal{P}}_r$ equals $2\,Tr(\mathbf{\Sigma}_{\tilde{\mathbf{Q}}})$.

Proof: The proof is similar to Theorem III.3. □

IV. Experimental Results

In this section, the new method (SPDE method) is applied to identify the joint space mass-inertia matrix, $\mathbf{M}$, of a University of Wyoming (UW) flexure jointed hexapod [3]. Flexure jointed hexapods are great candidates for micro-precision applications including micro-manipulation, laser weapon pointing, space-based interferometers, and optical communication, etc. Figure 2 shows the structure of the UW flexure jointed hexapod. Like any hexapod, it consists of a base attached to a base plate, a payload, and six struts (also called legs) connecting the payload to the base. Each

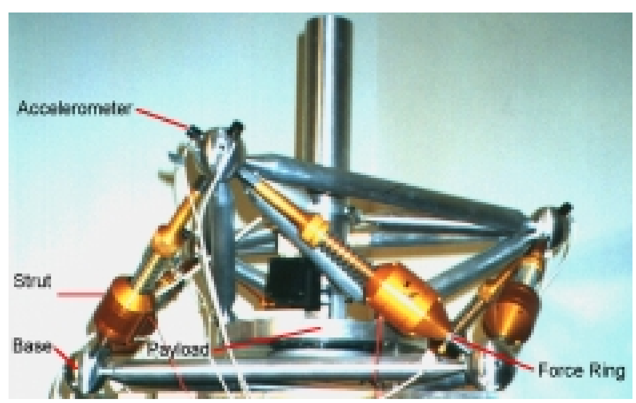

Fig. 2. University of Wyoming (UW) flexure jointed hexapod

strut contains springs which passively reduce vibrations from the base to the payload plate. A voice coil motor is also embedded into each strut. Thus the hexapod can slightly change the length of it legs to allow active vibration isolation and precise pose control of the payload.

In the control of the flexure jointed hexapod, the performance depends critically on the precision to which the decoupling matrix is calculated. The calculation is based on the joint space mass-inertia matrix [2], $\mathbf{M}$, of the hexapod. Although $\mathbf{M}$ can be calculated from the design parameters of the hexapod, it is laborious to do so and can introduce errors due to payload changes. Thus a better approach is to estimate $\mathbf{M}$ from the measured payload accelerations and base forces. The relationship between payload accelerations and base forces is described as $\mathbf{AM} \approx \mathbf{B}$ where $\mathbf{A}$ contains the payload accelerations, $\mathbf{B}$ contains the base forces, and there are sampling and instrument noises in both $\mathbf{A}$ and $\mathbf{B}$. The matrix data given below is calculated from the design parameters of UW's flexure jointed hexapod [3].

$$\mathbf{M} = \begin{bmatrix} 4.688 & 0.198 & -0.404 & 1.798 & -0.405 & 0.611 \\ 0.198 & 4.688 & 0.611 & -0.405 & 1.798 & -0.404 \\ -0.404 & 0.611 & 4.688 & 0.198 & -0.404 & 1.798 \\ 1.798 & -0.405 & 0.198 & 4.688 & 0.611 & -0.404 \\ -0.405 & 1.798 & -0.404 & 0.611 & 4.688 & 0.198 \\ 0.611 & -0.404 & 1.798 & -0.404 & 0.198 & 4.688 \end{bmatrix}.$$

For the identification experiment, a 200 MHz Gateway 2000 Pentium II computer running the QNX real time operating system sends commands, which are six independent random-noise signals filtered by six digital low-pass filters with cutoff frequencies at 80Hz, through Computers Boards 16 bit DAC converters to Techron linear current amplifiers. These activate BEI voice coil actuators to generate vibrations, resulting in base forces measured by PCB quartz force rings mounted on the bottom end of each strut. Payload accelerations are simultaneously measured by KISTLER-BEAM accelerometers mounted on the top spherical joints. The force and acceleration signals are sampled by the control computer's 16 bit A/D converters at a rate of 5kHz. 256K samples (each sample includes 6 base forces and 6 payload accelerations) are collected. The least squares (LS) estimates [11] and the estimates using the new method (SPDE method) are compared. For both methods, 100 experiments were performed. Since $\mathbf{M}$ is a 6×6 symmetric matrix, it has 21 independent parameters. The absolute value of the estimation error's mean and standard deviation are shown in Figure 3. Compared to the LS method,

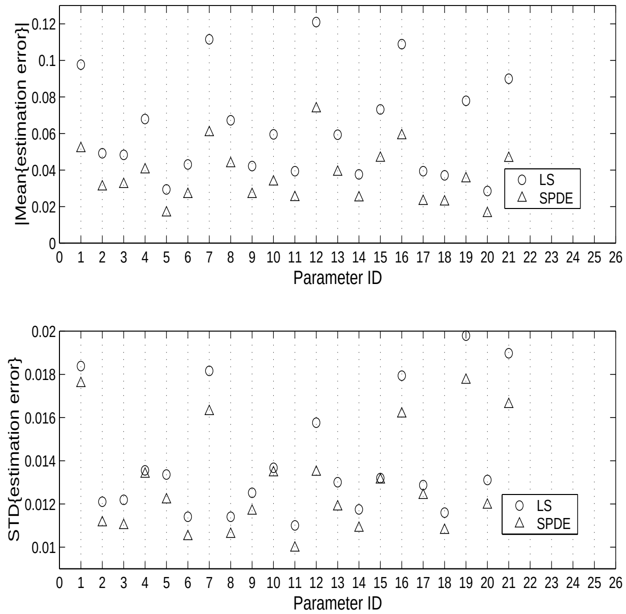

Fig. 3. Comparison of LS and SPDE methods: absolute mean and standard deviation of the estimation errors of $\mathbf{M}$'s entries. Mean{} and STD{} stand for the mean and the standard deviation, respectively.

the SPDE method provides more accurate estimates for all 21 parameters.

V. Conclusions

A new method of solving an over-determined set of linear equations $\mathbf{AX} \approx \mathbf{B}$ with a symmetric positive definite constraint and errors in both data matrices $\mathbf{A}$ and $\mathbf{B}$ is proposed. This type of problem arises in a number of contexts relevant to control problems, including estimation of mass-inertia and covariance matrices. The SPDE method transforms the original problem into an optimization problem seeking to minimize the so called area criterion. Compared with the least squares method, the new method improves the estimation accuracy because it takes errors in both $\mathbf{A}$ and $\mathbf{B}$ into consideration. In addition, no prior knowledge of the upper and lower bounds for the entries or eigenvalues of the fitting matrix are needed for the new method, which makes the new method easy to apply. Experiment results demonstrate that if the data matrices ($\mathbf{A}$ and $\mathbf{B}$) from both sides of the equation contain noise, then the new algorithm gives more precise estimates than the traditional least squares approach.

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[2] The joint space mass-inertia matrix does not depend on the coordinate frame attached to the payload. Details can be found in [3].

[3] We admit that there are errors in $\mathbf{M}$ due to discrepancies between real and design parameters. Compared with estimation errors, the errors induced by parameter differences are negligible because discrepancies are quite small. Thus we can safely assume $\mathbf{M}$ to be the real joint space mass-inertia matrix.

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