Notes on Models of Computation
Chapter 2

H. Conrad Cunningham

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Professor of Computer and Information Science
University of Mississippi
214 Weir Hall
P.O. Box 1848
University, MS 38677
(662) 915-7396 (dept. office)

Browser Advisory: The HTML version of this textbook requires a browser that supports the display of MathML. A good choice as of April 2022 is a recent version of Firefox from Mozilla.

Note: These notes were written primarily to accompany my use of Chapter 2 of the Linz textbook *An Introduction to Formal Languages and Automata* [[1].

2 Finite Automata

In chapter 2, we approach automata and languages more precisely and formally than in chapter 1.

A *finite automaton* is an automaton that has no temporary storage (and, like all automata we consider in this course, a finite number of states and input alphabet).

2.1 Deterministic Finite Accepters

2.1.1 Accepters

Linz Definition (DFA): A *deterministic finite accepter*, or *dfa*, is defined by the tuple $M = (Q, \Sigma, \delta, q_0, F)$ where

- Q is a finite set of *internal states*
- Σ is a finite set of symbols called the *input alphabet*
- $\delta : Q \times \Sigma \to Q$ is a total function called the *transition function*
- $q_0 \in Q$ is the *initial state.*
- $F \subseteq Q$ is a set of *final states.*

A dfa operates as described in the following pseudocode:

```
currentState := $q_0$

position input at left end of string

while more input exists currentInput := next_input_symbol
    advance input to right
    currentState := $\delta$(currentState,currentInput)

if currentState $\in F$ then ACCEPT else REJECT
```

2.1.2 Transition Graphs

To visualize a dfa, we use a *transition graph* constructed as follows:

- Vertices represent states
  - labels are state names
  - exactly one vertex for every $q_i \in Q$
- Directed edges represent transitions
  - label on edge is current input symbol
  - directed edge (q, r) with label a if and only if $\delta(q, a) = r$

2.1.3 Linz Example 2.1

The graph pictured in Linz Figure 2.1 represents the dfa $M = (\{q_0, q_1, q_2\}, \{0, 1\}, \delta, q_0, \{q_1\})$, where δ is represented by

$\delta(q_0, 0) = q_0,\ \delta(q_0, 1) = q_1$
$\delta(q_1, 0) = q_0,\ \delta(q_1, 1) = q_2$
$\delta(q_2, 0) = q_2,\ \delta(q_2, 1) = q_1$

Note that q_0 is the *initial state* and q_1 is the only *final state* in this dfa.

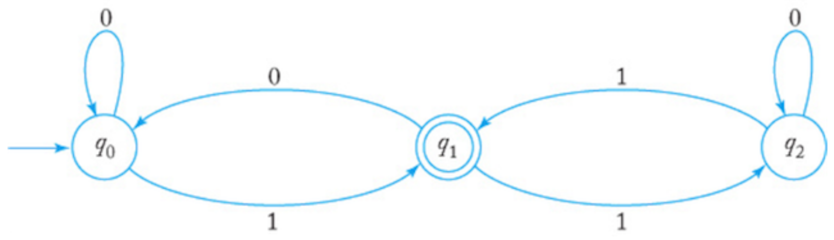


Figure 1: **Linz Fig. 2.1: DFA Transition Graph**

The dfa in Linz Figure 2.1:

- Accepts 01, 101
- Rejects 00, 100

What about 0111? 1100?

2.1.4 Extended Transition Function for a DFA

Linz Definition: The *extended transition function* $\delta^* : Q \times \Sigma^* \to Q$ is defined recursively:

$\delta^*(q, \lambda) = q$
$\delta^*(q, wa) = \delta(\delta^*(q, w), a)$

The extended transition function gives the state of the automaton after reading a string.

2.1.5 Language Accepted by a DFA

Linz Definition 2.2 (Language Accepted by DFA): The language accepted by a dfa $M = (Q, \Sigma, \delta, q_0, F)$ is $L(M) = \{w \in \Sigma^* : \delta^*(q_0, w) \in F\}$.

That is, $L(M)$ is the set of all strings on the input alphabet accepted by automaton M.

Note that above δ and δ^* are total functions (i.e., defined for all strings).

2.1.6 Linz Example 2.2

The automaton in Linz Figure 2.2 accepts all strings consisting of arbitrary numbers of a's followed by a single b.

In set notation, the language accepted by the automaton is $L = \{a^n b : n \geq 0\}$.

Note that q_2 has two self-loop edges, each with a different label. We write this compactly with multiple labels.

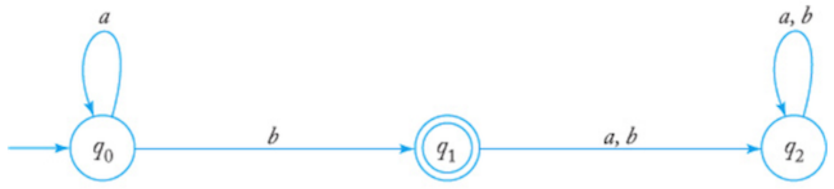


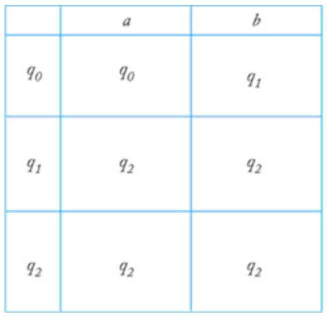
Figure 2: **Linz Fig. 2.2: DFA Transition Graph with Trap State**

A *trap state* is a state from which the automaton can never “escape”.

Note that q_2 is a trap state in the dfa transition graph shown in Linz Figure 2.2.

Transition graphs are quite convenient for understanding finite automata.

For other purposes–such as representing finite automata in programs–a tabular representation of transition function δ may also be convenient (as shown in Linz Fig. 2.3).



	a	b
q_0	q_0	q_1
q_1	q_2	q_2
q_2	q_2	q_2

Figure 3: **Linz Fig. 2.3: DFA Transition Table**

2.1.7 Linz Example 2.3

Find a deterministic finite accepter that recognizes the set of all string on $\Sigma = \{a, b\}$ starting with the prefix ab.

Linz Figure 2.4 shows a transition graph for a dfa for this example.

The dfa must accept ab and then continue until the string ends.

This dfa has a *final* trap state q_2 (accepts) and a *non-final* trap state q_3 (rejects).

2.1.8 Linz Example 2.4

Find a dfa that accepts all strings on $\{0, 1\}$, except those containing the substring 001.

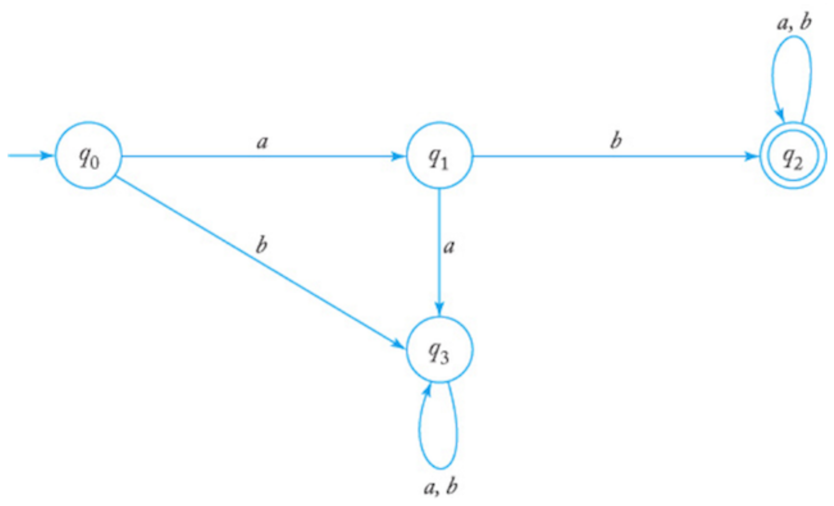


Figure 4: **Linz Fig. 2.4**

- need to "remember" whether last two inputs were 00
- use state changes for "memory"

Linz Figure 2.5 shows a dfa for this example.

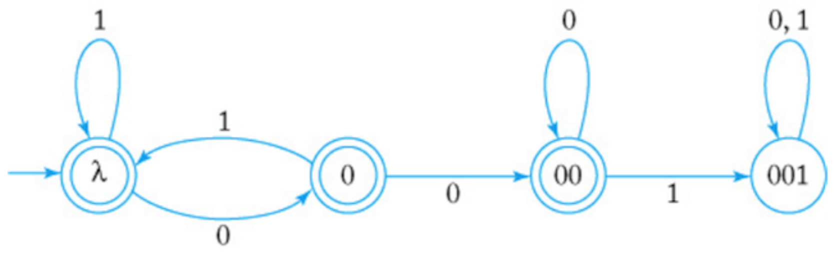


Figure 5: **Linz Fig. 2.5**

Accepts: λ, 0, 00, 01, 010000

Rejects: 001, 000001, 0010101010101

2.1.9 Regular Languages

Linz Definition 2.3 (Regular Language): A language L is called *regular* if and only if there exists a dfa M such that $L = L(M)$.

Thus dfas define the *family* of languages called *regular*.

2.1.10 Linz Example 2.5

Show that the language $L = \{awa : w \in \{a,b\}^*\}$ is regular.

- Construct a dfa.
- Check whether begin/end with "a".
- Am in final state when second a input.

Linz Figure 2.6 shows a dfa for this example.

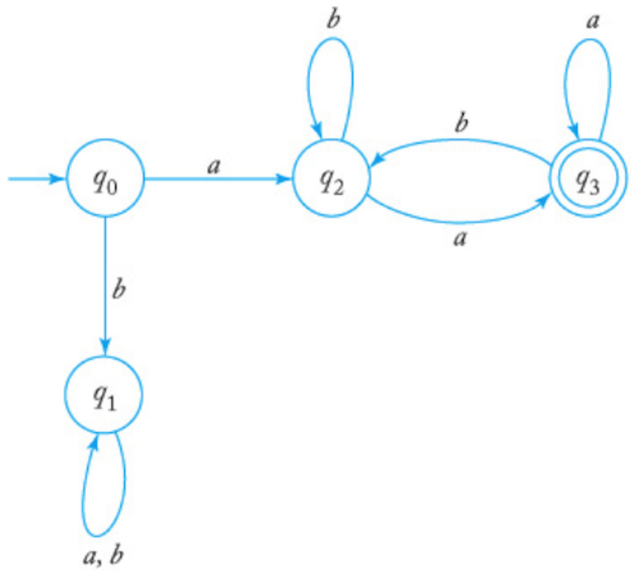


Figure 6: **Linz Fig. 2.6**

Question: How would we prove that a languages is not regular?

We will come back to this question in chapter 4.

2.1.11 Linz Example 2.6

Let L be the language in the previous example (Linz Example 2.5).

Show that L^2 is regular.

$L^2 = \{aw_1aaw_2a : w_1, w_2 \in \{a,b\}^*\}$.

- Construct a dfa.
- Use Example 2.5 dfa as starting point.
- Accept two consecutive strings of form awa.
- Note that any two consecutive a's could start a second string.

Linz Figure 2.7 shows a dfa for this example.

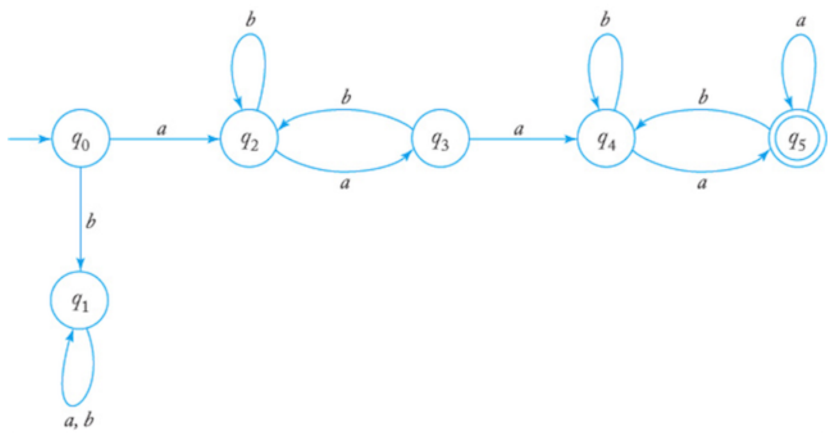


Figure 7: **Linz Fig. 2.7**

The last example suggests the conjecture that if a language L then so is L^2, L^3, etc.

We will come back to this issue in chapter 4.

2.2 Nondeterministic Finite Accepters

2.2.1 Nondeterministic Accepters

Linz Definition 2.4 (NFA): A *nondeterministic finite accepter* or *nfa* is defined by the tuple $M = (Q, \Sigma, \delta, q_0, F)$ where Q, Σ, q_0, and F are defined as for deterministic finite accepters, but

$$\delta : Q \times (\Sigma \cup \{\lambda\}) \rightarrow 2^Q.$$

Remember for dfas:

- Q is a finite set of *internal states*.
- Σ is a finite set of symbols called the *input alphabet*.
- $q_0 \in Q$ is the *initial state*.
- $F \subseteq Q$ is a set of *final states*.

The key differences between dfas and nfas are

1. dfa: δ yields a single state
   nfa: δ yields a set of states
2. dfa: consumes input on each move
   nfa: can move without input (λ)

3. dfa: moves for all inputs in all states
   nfa: some situations have no defined moves

An nfa *accepts* a string if *some possible* sequence of moves ends in a final state.

An nfa *rejects* a string if *no possible* sequence of moves ends in a final state.

2.2.2 Linz Example 2.7

Consider the transition graph shown in Linz Figure 2.8.

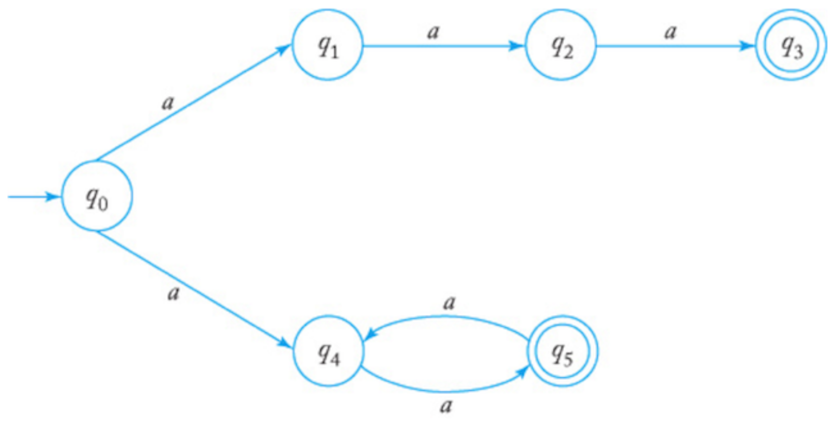


Figure 8: **Linz Fig. 2.8**

Note the nondeterminism in state q_0 with two possible transitions for input a.

Also state q_3 has no transition for any input.

2.2.3 Linz Example 2.8

Consider the transition graph for an nfa shown in Linz Figure 2.9.

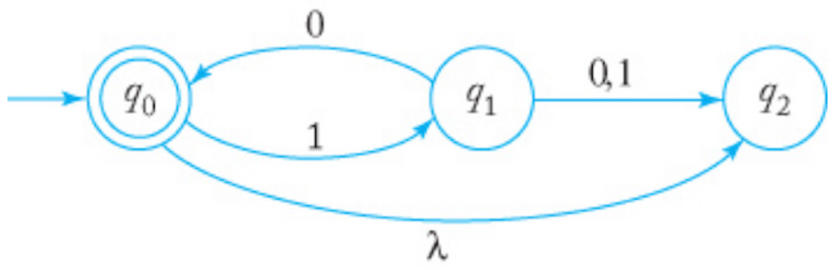


Figure 9: **Linz Fig. 2.9**

Note the nondeterminism and the λ-transition.

Note: Here λ means the move takes place without consuming any input symbol. This is different from accepting an empty string.

Transitions:

- for $(q_0, 0)$?
- for $(q_1, 0)$?
- for $(q_2, 0)$?
- for $(q_2, 1)$?

Accepts: λ, 10, 1010, 101010

Rejects: 0, 11

2.2.4 Extended Transition Function for an NFA

As with dfas, the transition function can be *extended* so its second argument is a string.

Requirement: $\delta^*(q_i, w) = Q_j$ where Q_j is the set of all possible states the automaton may be in, having started in state q_i and read string w.

Linz Definition (Extended Transition Function): For an nfa, the *extended transition function* is defined so that $\delta^*(q_i, w)$ contains q_j if there is a walk in the transition graph from q_i to q_j labelled w.

2.2.5 Language Accepted by an NFA

Linz Definition 2.6 (Language Accepted by NFA): The language L accepted by the nfa $M = (Q, \Sigma, \delta, q_0, F)$ is defined

$$L(M) = \{w \in \Sigma^* : \delta^*(q_0, w) \in F \neq \emptyset\}.$$

That is, $L(M)$ is the set of all strings w for which there is a walk labeled w from the initial vertex of the transition graph to some final vertex.

2.2.6 Linz Example 2.10 (Example 2.8 Revisited)

Let's again examine the automaton given in Linz Figure 2.9 (Example 2.8).

This nfa, call it M:

- must end in q_0
- $L(M) = \{(10)^n : n \geq 0\}$

Note that q_2 is a *dead configuration* because $\delta^*(q_0, 110) = \emptyset$.

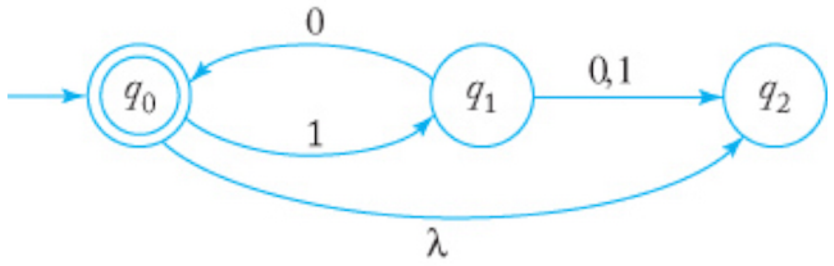


Figure 10: **Linz Fig. 2.9 (Repeated)**

2.2.7 Why Nondeterminism

When computers are deterministic?

- an nfa can model a search or backtracking algorithm
- nfa solutions may be simpler than dfa solutions (can convert from nfa to dfa)
- nondeterminism may model externally influenced interactions (or abstract more detailed computations)

2.3 Equivalence of DFAs and NFAs

2.3.1 Meaning of Equivalence

When are two mechanisms (e.g., programs) equivalent?

- When they have exactly the same descriptions?
- When they always go through the exact same sequence of steps?
- When the same input generates the same output for both?

The last seems to be the best approach.

Linz Definition 2.7 (DFA-NFA Equivalence): Two finite accepters M_1 and M_2 are said to be *equivalent* if $L(M_1) = L(M_2)$. That is, if they both accept the same language.

Here "same language" refers to the *input* and "both accept" refers to the *output*.

Often there are many accepters for a language.

Thus there are many equivalent dfa and nfas.

2.3.2 Linz Example 2.11

Again consider the nfa represented by the graph in Linz Fig. 2.9. Call this M_1.

As we saw, $L(M_1) = \{(10)^n : n \geq 0\}$.

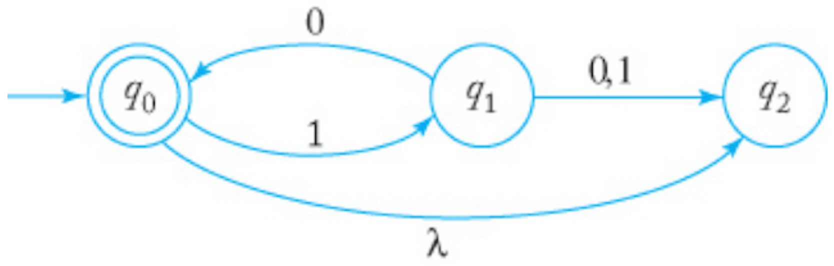


Figure 11: **Linz Fig. 2.9 (Repeated): An NFA**

Now consider the dfa represented by the graph in Linz Figure 2.11. Call this M_2.

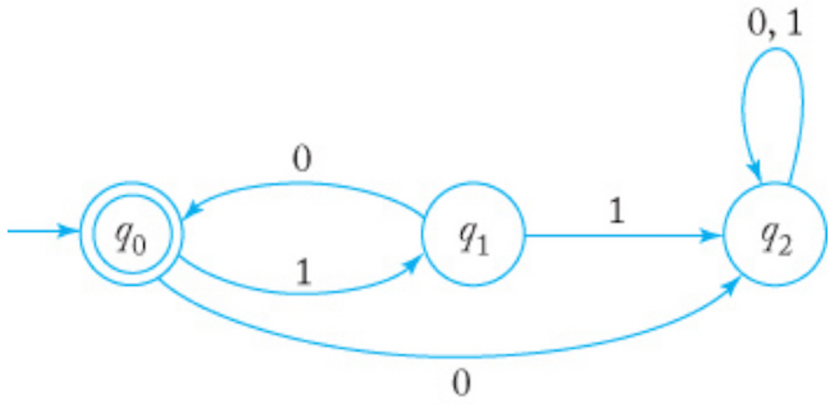


Figure 12: **Linz Fig. 2.11: DFA Equivalent to Fig. 9 NFA**

$L(M_2) = \{(10)^n : n \geq 0\}$.

Thus, M_1 is equivalent to M_2.

2.3.3 Power of NFA versus DFA

Which is more powerful, dfa or nfa?

Clearly, any dfa D can be made into a nfa N.

- Keep the same states.
- Define $\delta_N(q, a) = \{\delta_D(q, a)\}$.

Can any nfa N be made into a dfa D?

- Yes, but it is less obvious. (See theorem below.)

Thus, dfas and nfas are “equally powerful”.

Linz Theorem 2.2 (Existence of DFA Equivalent to NFA): Let L be the language accepted by the nfa $M_N = (Q_N, \Sigma, \delta_N, q_0, F_N)$. Then there exists a dfa $M_D = (Q_D, \Sigma, \delta_D, \{q_0\}, F_D)$ such that $L = L(M_D)$.

A pure mathematician would be content with an existence proof.

But, as computing scientists, we want an *algorithm* for construction of M_D from M_N. The proof of the theorem follows from the correctness of the following algorithm.

Key ideas:

- After reading w, M_N will be in the some state from $\{q_i, q_j, \ldots\ q_k\}$. That is, $\delta^*(q_0, w) = \{q_i, q_j, \ldots\ q_k\}$.
- Label the *dfa state* that has accepted w with the set of nfa states $\{q_i, q_j, \ldots\ q_k\}$. This is an interesting "trick"!

Remember from the earlier definitions in these notes and from discrete mathematics:

- Σ is finite (and the same for the nfa and dfa).
- Q_N is finite.
- δ_D is a total function. That is, every vertex of the dfa graph has $|\Sigma|$ outgoing edges.
- The maximum number of dfa states with the above labeling is $|2^{Q_N}| = 2^{|Q_N|}$. Hence, finite.
- The maximum number of dfa edges is $2^{|Q_D|}|\Sigma|$. Hence, finite.

Procedure nfa_to_dfa

Given a transition graph G_N for nfa $M_N = (Q_N, \Sigma, \delta_N, q_0, F_N)$, construct a transition graph G_D for a dfa $M_D = (Q_D, \Sigma, \delta_D, q_0, F_D)$. Label each vertex in G_D with a *subset* of the vertices in G_N.

1. Initialize graph G_D to have an initial vertex $\{q_0\}$ where q_0 is the initial state of G_N.
2. Repeat the following steps until no more edges are missing from G_D:
    a. Take any vertex from G_D labeled $\{q_i, q_j, \ldots\ q_k\}$ that has no outgoing edge for some $a \in \Sigma$.
    b. For this vertex and input, compute $\delta_N^*(q_i, a), \delta_N^*(q_j, a), \ldots\ \delta_N^*(q_k, a)$. (Each of these is a set of states from Q_N.)
    c. Let $\{q_l, q_m, \ldots\ q_n\}$ be the union of all δ_N^* sets formed in the previous step.
    d. If vertex $\{q_l, q_m, \ldots\ q_n\}$ constructed in the previous step (step 2c) is not already in G_D, then add it to G_D.

   e. Add an edge to G_D from vertex $\{q_i, q_j, \ldots\ q_k\}$ (vertex selected in step 2b) to vertex $\{q_l, q_m, \ldots\ q_n\}$ (vertex possibly created in step 2d) and label the new edge with a (input selected in step 2b).

3. Make every vertex of G_D whose label contains any vertex $q_f \in F_N$ a final vertex of G_D.
4. If M_N accepts λ, then vertex $\{q_0\}$ in G_D is also a final vertex.

This, if the loop terminates, it constructs the dfa corresponding to the nfa.

Does the loop terminate?

- Each iteration of the loop adds one edge to G_D.
- There are a finite number of edges possible in G_D.
- Thus the loop must terminate.

What is the loop invariant? (This ia a property always that must hold at the loop test.)

- If there is a walk $(\{q_0\}, \ldots\ \{\ldots\ q_i, \ldots\})$ in G_D labeled w, then there is a walk $(q_0, \ldots\ q_i)$ in G_N labeled w.

2.3.4 Linz Example 2.12

Convert the nfa in Linz Figure 2.12 to an equivalent dfa.

Intermediate steps are shown in Figures 2.12-1 and 2.12-2, with the final results in Linz Figure 2.13.

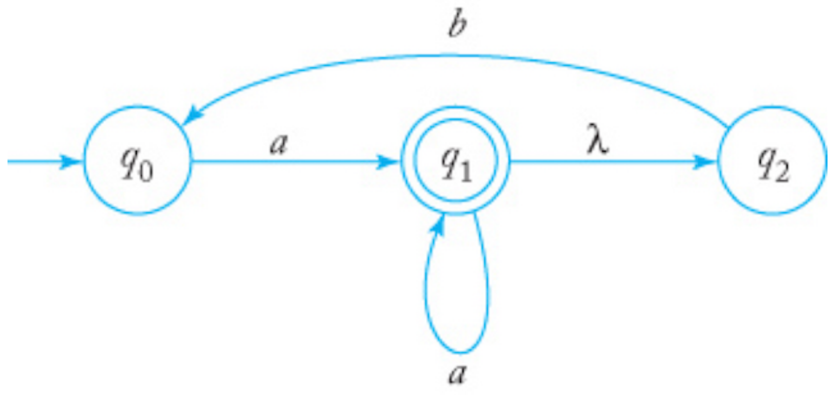


Figure 13: **Linz Fig. 2.12: An NFA**

2.3.5 Linz Example 2.13

Convert the nfa shown in Linz Figure 2.14 into an equivalent dfa.

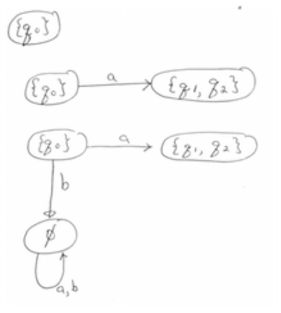


Figure 14: **Intermediate Fig. 2.12-1**

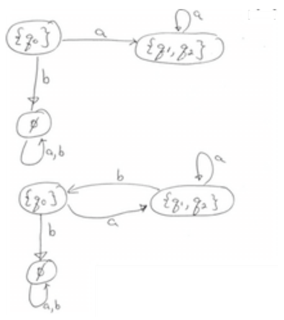


Figure 15: **Intermediate Fig. 2.12-2**

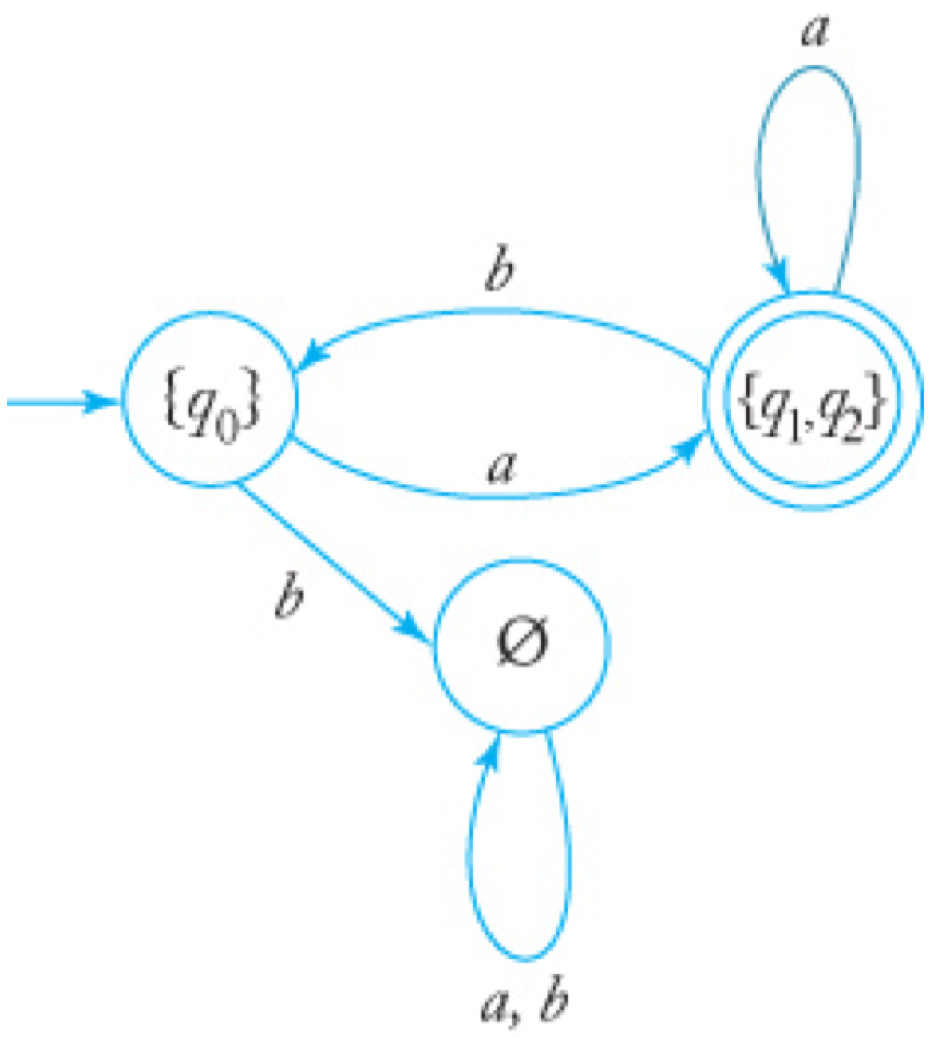


Figure 16: **Linz Fig. 2.13: Corresponding DFA**

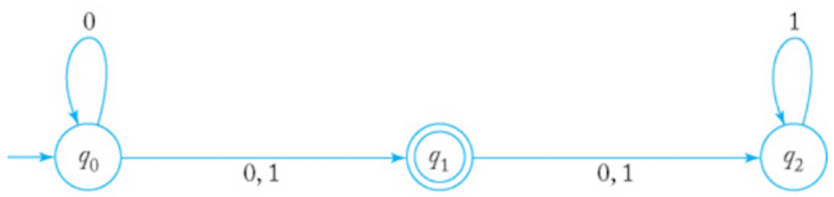


Figure 17: **Linz Fig. 2.14: An NFA**

$\delta_D(\{q_0\}, 0) = \delta_N^*(q_0, 0) = \{q_0, q_1\}$

$\delta_D(\{q_0\}, 1) = \delta_N^*(q_0, 1) = \{q_1\}$

$\delta_D(\{q_0, q_1\}, 0) = \delta_N^*(q_0, 0) \cup \delta_N^*(q_1, 0) = \{q_0, q_1, q_2\}$

$\delta_D(\{q_0, q_1, q_2\}, 1) = \delta_N^*(q_0, 1) \cup \delta_N^*(q_1, 1) \cup \delta_N^*(q_2, 1) = \{q_1, q_2\}$

The above gives us the partially constructed dfa shown in Linz Figure 2.15.

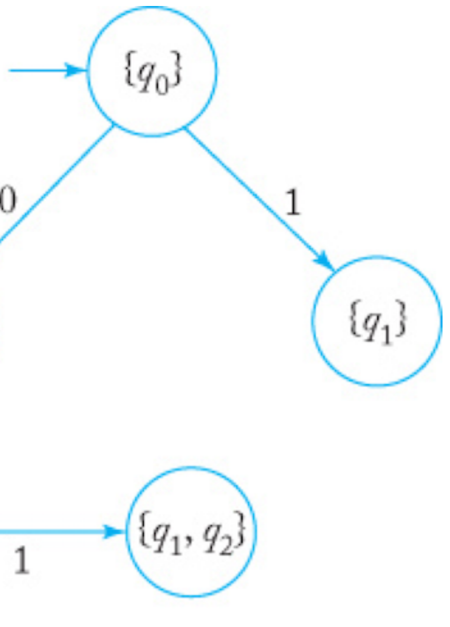


Figure 18: **Linz Fig. 2.15**

$\delta_D(\{q_1\}, 0) = \delta_N^*(q_1, 0) = \{q_2\}$

$\delta_D(\{q_1\}, 1) = \delta_N^*(q_1, 1) = \{q_2\}$

$\delta_D(\{q_2\}, 0) = \delta_N^*(q_1, 0) = \emptyset$

$\delta_D(\{q_2\}, 1) = \delta_N^*(q_2, 1) = \{q_2\}$

$\delta_D(\{q_0, q_1\}, 1) = \delta_N^*(q_0, 1) \cup \delta_N^*(q_1, 1) = \{q_1, q_2\}$

$\delta_D(\{q_0, q_1, q_2\}, 0) = \delta_N^*(q_0, 0) \cup \delta_N^*(q_1, 0) \cup \delta_N^*(q_2, 0) = \{q_0, q_1, q_2\}$

$\delta_D(\{q_1, q_2\}, 0) = \delta_N^*(q_1, 0) \cup \delta_N^*(q_2, 0) = \{q_2\}$

$\delta_D(\{q_1, q_2\}, 1) = \delta_N^*(q_1, 1) \cup \delta_N^*(q_2, 1) = \{q_2\}$

Now, the above gives us the dfa shown in Linz Figure 2.16.

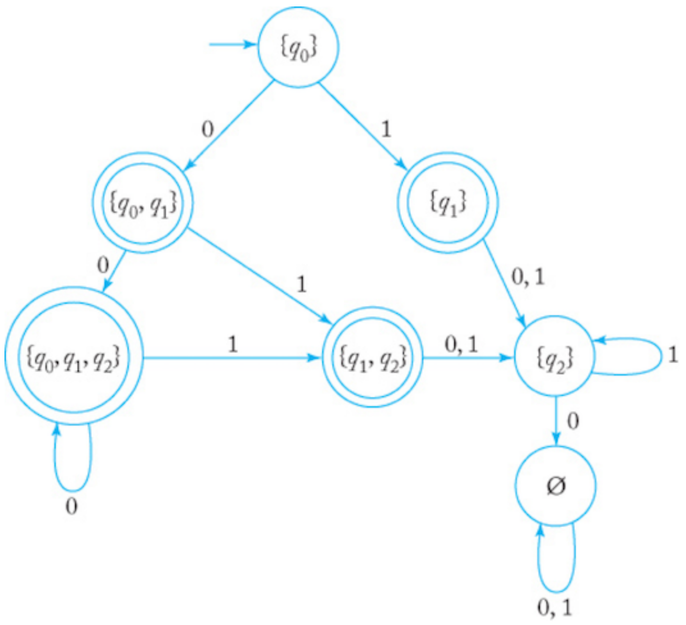


Figure 19: **Linz Fig. 2.16: Corresponding DFA for NFA**

2.4 Reduction in the Number of States in Finite Automata

This section is not covered in this course.

2.5 References

[1] Peter Linz. 2011. *Formal languages and automata* (Fifth ed.). Jones & Bartlett, Burlington, Massachusetts, USA.