# Exploring Languages with Interpreters and Functional Programming

Chapter 24

H. Conrad Cunningham

25 July 2018

## Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>24 Type Inference</td>
<td>2</td>
</tr>
<tr>
<td>24.1 Chapter Introduction</td>
<td>2</td>
</tr>
<tr>
<td>24.2 Motivation</td>
<td>2</td>
</tr>
<tr>
<td>24.3 Example: Functional Composition</td>
<td>2</td>
</tr>
<tr>
<td>24.4 Example: Multiple Use of Polymorphic Function (fst)</td>
<td>3</td>
</tr>
<tr>
<td>24.5 Example: fix</td>
<td>5</td>
</tr>
<tr>
<td>24.6 Example: Incorrect Typing (selfapply)</td>
<td>6</td>
</tr>
<tr>
<td>24.7 Other Aspects of Type Inference</td>
<td>7</td>
</tr>
<tr>
<td>24.8 What Next?</td>
<td>7</td>
</tr>
<tr>
<td>24.9 Exercises</td>
<td>7</td>
</tr>
<tr>
<td>24.10 Acknowledgements</td>
<td>7</td>
</tr>
<tr>
<td>24.11 References</td>
<td>8</td>
</tr>
<tr>
<td>24.12 Terms and Concepts</td>
<td>8</td>
</tr>
</tbody>
</table>

Copyright (C) 2017, 2018, H. Conrad Cunningham
Professor of Computer and Information Science
University of Mississippi
211 Weir Hall
P.O. Box 1848
University, MS 38677
(662) 915-5358

**Browser Advisory:** The HTML version of this textbook requires a browser that supports the display of MathML. A good choice as of July 2018 is a recent version of Firefox from Mozilla.
24 Type Inference

24.1 Chapter Introduction

The goal of this chapter is to show how type inference works. It presents the topic using an equational reasoning technique.

This chapter depends upon the reader understanding Haskell polymorphic, higher-order function concepts (e.g. from studying Chapters 13-17), but it is otherwise independent of others chapters. No subsequent chapter depends explicitly upon this content.

24.2 Motivation

How can we deduce the type of a Haskell expression?

To get the general idea, let’s look at a few examples.

Note: The discussion here is correct for monomorphic functions, but it is a bit simplistic for polymorphic functions. However, it should be of assistance in understanding how types are assigned to Haskell expressions.

24.3 Example: Functional Composition

Expressed in prefix form, functional composition can be defined with the equation:

\( (\cdot) \ f \ g \ x = f \ (g \ x) \)

We begin the process of type inference by assigning types to the parameter names and to the function’s defining expression (i.e. its result). We introduce new type names \( t_1, t_2, t_3 \) and \( t_4 \) for the components of \( (\cdot) \) as follows:

\[
\begin{align*}
  f & :: t_1 & \quad \text{- parameter of } (\cdot) \\
  g & :: t_2 & \quad \text{- parameter of } (\cdot) \\
  x & :: t_3 & \quad \text{- parameter of } (\cdot) \\
  f \ (g \ x) & :: t_4 & \quad \text{- defining expression for } (\cdot)
\end{align*}
\]

The type of \( (\cdot) \) is therefore given by:

\( (\cdot) :: t_1 \rightarrow t_2 \rightarrow t_3 \rightarrow t_4 \)

We are not finished because there are certain relationships among the new types that must be taken into account. To see what these relationships are, we use the following inference rules.

- **Application rule:** If \( f \ x :: t \), then we can deduce \( x :: t' \) and \( f :: t' \rightarrow t \) for some new type \( t' \).
• **Equality rule:** If both \( x :: t \) and \( x :: t' \) for some variable \( x \), then we can deduce \( t = t' \).

• **Function rule:** If \( (t \to u) = (t' \to u') \), then we can deduce \( t = t' \) and \( u = u' \).

Using the *application rule* on \( f \ (g \ x) :: t4 \), we introduce a new type \( t5 \) such that:

\[
\begin{align*}
g \ x & :: t5 \\
f & :: t5 \to t4
\end{align*}
\]

Using the *application rule* for \( g \ x :: t5 \), we introduce another new type \( t6 \) such that:

\[
\begin{align*}
x & :: t6 \\
g & :: t6 \to t5
\end{align*}
\]

Using the *equality rule* on the two types deduced for each of \( f \), \( g \), and \( x \), respectively, we get the following identities:

\[
\begin{align*}
t1 & = (t5 \to t4) \quad -- f \\
t2 & = (t6 \to t5) \quad -- g \\
t3 & = t6 \quad -- x
\end{align*}
\]

For function \( (\_ \_ \_ ) \), we thus deduce type signature:

\[
(\_ \_ \_ ) :: (t5 \to t4) \to (t6 \to t5) \to t6 \to t4
\]

If we replace the type names by Haskell generic type variables that follow the usual naming convention, we get:

\[
(\_ \_ \_ ) :: (b \to c) \to (a \to b) \to a \to c
\]

### 24.4 Example: Multiple Use of Polymorphic Function (*fst*)

Now let’s consider the function definition:

\[
f \ x \ y = \text{fst} \ x + \text{fst} \ y
\]

Note that the names \((+)\) and \(\text{fst}\) occur on the right side of the definition, but do not occur on the left.

From the Haskell Prelude, we can see that:

\[
\begin{align*}
(+) & :: \text{Num} \ a \Rightarrow a \to a \to a \\
\text{fst} & :: (a, b) \to a
\end{align*}
\]

The \text{Num} a context constrains the polymorphism on type variable \( a \).
We must be careful. The two occurrences of the polymorphic function `fst` in the definition for `f` need not bind the type variables `a` and `b` to the same concrete types. For example, consider the expression:

```haskell
fst (2, True) + fst (1, "hello")
```

This expression is well-typed despite the fact that the first occurrence of `fst` has the type

```haskell
Num a => (a,Bool) -> a
```

and the second occurrence has type

```haskell
Num a => (a, [Char]) -> a
```

Furthermore, the two occurrences of the type variable `a` are not, in general, required to bind to the same type. (However, as we will see, they do in this expression because of the addition operation.)

To handle the situation with the multiple applications of `fst`, we use the following rule.

- **Polymorphic use rule:** If a polymorphic function is applied multiple times in an expression, then the type of each occurrence is determined independently, with each assigned new type variables.

Following the polymorphic use rule, we rewrite the definition of `f` in the form

```haskell
f x y = fst1 x + fst2 y
```

and assume two different instantiations of the generic type of `fst`:

- `fst1 :: (u1, u2) -> u1`
- `fst2 :: (v1, v2) -> v1`

After making the above transformation, we proceed by assigning types to the parameters and definition of `f`, introducing three new types:

- `x :: t1` -- parameter of `f`
- `y :: t2` -- parameter of `f`
- `fst1 x + fst2 y :: t3` -- defining expression for `f`

Thus we have the following type for `f`:

```haskell
f :: t1 -> t2 -> t3
```

Now we can rewrite the defining expression for `f` fully in prefix form to get:

```haskell
(+)(fst1 x)(fst2 y)
```

Then, using the application rule on the above expression, we deduce:

- `(fst2 y) :: t4`
- `(+)(fst1 x) :: t4 -> t3`

Using the application rule on `(fst2 y) :: t4`, we get:
Similarly, using the application rule on \((+)\) \((\text{fst1 } x)\) :: \(t4 \rightarrow t3\), we get:

\[
(\text{fst1 } x) :: t6 \\
(+) :: t6 \rightarrow t4 \rightarrow t3
\]

Going further and applying the application rule to \((\text{fst1 } x)\) :: \(t6\), we deduce:

\[
x :: t7 \\
\text{fst1} :: t7 \rightarrow t6
\]

Now we have introduced types for all the symbols appearing in the definition of function \(f\). We begin simplification by using the equality rule for \(x\), \(y\), \(\text{fst1}\), \(\text{fst2}\), and \((+)\), respectively. We thus deduce the type equations:

\[
t1 = t7 \quad -- x \\
t2 = t5 \quad -- y \\
((u1, u2) \rightarrow u1) = (t7 \rightarrow t6) \quad -- \text{fst1} \\
((v1, v2) \rightarrow v1) = (t5 \rightarrow t4) \quad -- \text{fst2} \\
(\text{Num } a \Rightarrow a \rightarrow a \rightarrow a) = (t6 \rightarrow t4 \rightarrow t3) \quad -- (+)
\]

Now, using the function rule on the last three equations above, we derive:

\[
t7 = (u1, u2) \\
t6 = u1 \\
t5 = (v1, v2) \\
t4 = v1 \\
t3 = t4 = t6 = v1 = u1 = (\text{Num } a \Rightarrow a)
\]

We had assigned type \(f :: t1 \rightarrow t2 \rightarrow t3\) originally. Substituting from the above, we deduce the following type:

\[
f :: \text{Num } a \Rightarrow (a, u2) \rightarrow (a, v2) \rightarrow a
\]

Finally, we can replace the type names \(u2\) and \(v2\) by Haskell generic type variables that follow the usual naming convention. We get the following inferred type for function \(f\):

\[
f :: \text{Num } a \Rightarrow (a, b) \rightarrow (a, c) \rightarrow a
\]

### 24.5 Example: \texttt{fix}

For this example, consider the definition:

\[
\text{fix } f = f (\text{fix } f)
\]

To deduce a type for \texttt{fix}, we proceed as before and introduce types for the parameters and defining expression of \(f\):
Thus, fix has the type:

\[
\text{fix} :: t_1 \rightarrow t_2
\]

Using the application rule on the expression \( f \ (\text{fix} \ f) \), we obtain:

\[
\begin{align*}
(\text{fix} \ f) & :: t_3 \\
f & :: t_3 \rightarrow t_2
\end{align*}
\]

Then using the application rule on the expression \( \text{fix} \ f \), we get:

\[
\begin{align*}
f & :: t_4 \\
\text{fix} & :: t_4 \rightarrow t_3
\end{align*}
\]

Using the equality rule on \( f \) and \( \text{fix} \), we deduce:

\[
\begin{align*}
t_1 &= t_4 = (t_3 \rightarrow t_2) & -- f \\
(t_1 \rightarrow t_2) &= (t_4 \rightarrow t_3) & -- \text{fix}
\end{align*}
\]

Then, using the function rule on the second equation, we obtain the identities:

\[
\begin{align*}
t_1 &= t_4 \\
t_2 &= t_3
\end{align*}
\]

Since \( \text{fix} :: t_1 \rightarrow t_3 \), we derive the type:

\[
\text{fix} :: (t_3 \rightarrow t_3) \rightarrow t_3
\]

If we replace \( t_3 \) by a Haskell generic type variable that follows the usual naming convention, we get the following inferred type for \( \text{fix} \):

\[
\text{fix} :: (a \rightarrow a) \rightarrow a
\]

### 24.6 Example: Incorrect Typing (selfapply)

Finally, let us consider an example in which the typing is wrong. Let us define \( \text{selfapply} \) as follows:

\[
\text{selfapply} \ f = f \ f
\]

Proceeding as in the previous examples, we introduce new types for the parameters and defining expression of \( f \):

\[
\begin{align*}
f & :: t_1 & -- \text{parameter of selfapply} \\
f \ f & :: t_2 & -- \text{defining expression for selfapply}
\end{align*}
\]

Thus we have the type:

\[
\text{selfapply} :: t_1 \rightarrow t_2
\]

Using the application rule on \( f \ f \), we get:
\begin{verbatim}
  f :: t3
  f :: t3 -> t2
\end{verbatim}

But the equality rule for \( f \) tells us that:
\[
  t_1 = t_3 = (t_3 \rightarrow t_2)
\]
or just
\[
  t_1 = (t_1 \rightarrow t_2)
\]

However, the equation \( t_1 = (t_1 \rightarrow t_2) \) does not possess a solution for \( t_1 \) and the definition of \texttt{selfapply} is thus rejected by the type checker.

### 24.7 Other Aspects of Type Inference

Haskell function definitions must also conform to the following rules.

- **Guard rule:** Each guard must be an expression of type \texttt{Bool}.
- **Tuple rule:** The type of a tuple of elements is the tuple of their respective types.

### 24.8 What Next?

This chapter is largely independent of others chapters. No subsequent chapter depends explicitly upon this content.

### 24.9 Exercises

TODO

### 24.10 Acknowledgements

In Spring 2017, I adapted and revised this chapter from my previous HTML notes on this topic. I based the previous notes on the presentations in:


In Summer 2018, I incorporated this work as new Chapter 24, Type Inference, in the 2018 version of the textbook \textit{Exploring Languages with Interpreters and Functional Programming} and continue to revise it.
I maintain this chapter as text in Pandoc’s dialect of Markdown using embedded \LaTeX{} markup for the mathematical formulas and then translate the document to HTML, PDF, and other forms as needed.

## 24.11 References

TODO: Update


## 24.12 Terms and Concepts

Type inference, function, polymorphism, type variable, function composition, fixpoint, application rule, equality rule, function rule, polymorphic use rule, guard rule, tuple rule.