A few good tutorial videos:

Some of the videos use different notation, but the ideas are the same. If you don’t understand the concepts from one example, hearing it explained a slightly different way can help.

- https://www.youtube.com/watch?v=LC0J45agGBU
- https://www.youtube.com/watch?v=BJQmV3oKqgo
- https://www.youtube.com/watch?v=oZcQQ5nHHXg

General outline of Pumping Lemma

**Statement:** I want to prove that some language \( L \) is not regular.

**Proof by contradiction:**

Assume \( L \) is regular.

Let \( m \) be some given number.

Let \( w \) be a string that is accepted by the language and is at least \( m \) characters long.

that is, \( |w| \geq m \)

since \( w \in L \) and \( |w| \geq m \), the pumping lemma must apply. Specifically,

\[ w = xyz \] where
by the pumping lemma, the decomposition $x'y'z$ of $w$, is also accepted $L$.

We try to find some value for $i$ that results in a string not accepted by $L$. If we do, we have found a contradiction and proven that $L$ is not regular.

**Strategy**

Given $m$, we should try to pick a string $w$ that forces a predictable decomposition $xyz$.

for example (Linz Example 4.9), if $L = \{w \in \Sigma^* : n_a(w) < n_b(w)\}$, then a good $w$ to pick would be $a^m b^{m+1}$.

This is the part that's hardest to understand: because of the string that we picked, and we know that by the pumping lemma, $|xy| \leq m$, the only valid decomposition of our string must be so that $xy$ consists entirely of $a$’s.

For example if our string is $a^3 b^{3+1}$, or $aaabbb$, then because $m = 3$, and the length of $xy$ cannot be greater than 3, we know that $xy$ must consist entirely of $a$’s. More specifically, $y$, which is the part of the string that will be pumped, must consist of only $a$’s.

So we have $y = a^k$ where $1 \leq k \leq m$.

Then we have to pick an $i$ so that the resulting string, $w_i$ is not part of the language.

Going back to Linz Example 4.9, we pick $i = 2$ so that the resulting string $w_2 = a^{m+k} b^{m+1}$ is a contradiction because $m + k \geq m + 1$.

In general there is no way to know if you should pump up ($i > 1$) or down ($i = 0$). You need to pick $w$ so that you have a predictable decomposition $xyz$ so that when you pump $y^i$, you know exactly what you will get, and it should be easy to find an $x'y'z$ that violates $L$. 


A Simple Example

$L = \{a^n b^n : n \in \mathbb{N}\}$

**Statement:** I want to prove $L$ is not regular.

**Proof:** Assume $L$ is regular.

Given some $m$,

\[
\text{let } w = a^m b^m
\]

Since $w \in L$ and $|w| \geq m$, the pumping lemma applies to this selection of $w$.

So, by the pumping lemma, we get

\[
w = xyz \text{ where } |y| \geq 1 \text{ and } |xy| \leq m.
\]

$y = a^k$ where $0 < k \leq m$

$x = a^q$ where $0 \leq q < m$

So $z$ is everything that isn’t in $xy$,

\[
z = a^{m-k} b^m
\]

choose $i = 2$,

\[
w_2 = xy^2z = xyyz = a^q a^k a^{m-k-q} b^m
\]

\[
= a^{q+k+m-k-q} b^m
\]

\[
= a^{m+k} b^m, \text{ which is a contradiction because } m + k \neq m
\]